

Modeling Vocal Fold Motion with a Continuum Fluid Dynamic Model: I. Derivation and Analysis.

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Abstract

Vocal fold (VF) motion is fundamental to voice production and diagnosis in speech and health sciences. The motion is a consequence of air flow interacting with elastic vocal fold structures. Motivated by existing lumped mass models and known flow properties, we propose to model the continuous shape of vocal fold in motion by the two dimensional compressible Navier-Stokes equations coupled with an elastic damped driven wave equation on the fold cover. In this paper, instead of pursuing a direct two dimensional numerical simulation, we derive reduced quasi-one-dimensional model equations by averaging two dimensional solutions along the flow cross sections. We then analyze the oscillation modes of the linearized system about a flat fold, and found that the fold motion goes through a Hopf bifurcation into temporal oscillation if the flow energy is sufficient to overcome the damping in the fold consistent with the early models. We also analyze the further reduced system under the quasi-steady approximation and compare the resulting vocal fold equation in the small vibration regime with that of the Titze model. Our model shares several qualitative features with the Titze model yet differs in the specific form of energy input from the air flow to the fold. Numerical issues and results of the quasi-one-dimensional model system will be presented in part II (view resulting web VF animation at <http://www.ma.utexas.edu/users/jxin>).

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1 Introduction

Vocal folds consist of two lips of opposing ligaments and muscle at the top of the trachea and join to the lower vocal tract. When air is expelled at sufficient velocity through this orifice (the glottis) from the lung after a critical lung pressure is achieved, the folds vibrate and act as oscillating valves interrupting the airflow into a series of pulses. These pulses of air flow serve as the excitation source for the vocal tract in all voiced sounds. For background description of vocal folds and speech production, see [4], [5], [23] among others. Mathematical modeling of vocal fold motion can help us understand the voiced sound generation and make synthetic machine voice more natural. Such an understanding also helps the medical diagnosis and correction of voice disorders [8], [15].

Vocal fold modeling relies on our knowledge of aerodynamics and biomechanics, as the problem is basically airflow through a flexible channel with time-varying cross section. One of the best known models of vocal folds is the two mass model by Ishizaka and Flanagan [8]. The vocal fold is modeled by upper and lower masses (m_i , $i = 1, 2$) connected by an elastic spring and attached to the wall by upper and lower springs with damping. The two mass motion is given by the ODE system:

$$m_i x_{i,tt} + r_i x_{i,t} + k_i x_i + k_c (-1)^{i+1} (x_1 - x_2) = F_i, \quad i = 1, 2, \quad (1.1)$$

where r_i 's are the damping coefficients; k_i 's, k_c , are the elastic restoring constants (stiffness coefficients) for the upper, lower and side springs respectively; x_i 's are the displacements of two masses from their prephonatory equilibrium positions; F_i 's are driving forces acting on the two masses from glottal air pressure. To close the equations and complete the model, simplified assumptions are made to allow a calculation of F_i from the fluid flow so that $F_i = F_i(x_1, x_2, d_i, l_i, P_s)$, where P_s is the lung pressure, d_i the thickness of m_i along the flow direction, l_i the length of m_i transverse to the flow direction. The flow is assumed to be one dimensional, inviscid, and quasisteady so that

Bernoulli's law can be applied. Moreover, F_i and elastic forces $k_i x_i$ need to be modified empirically when the vocal folds tend to a closure to facilitate a pressure buildup to reopen the folds. In spite of these oversimplifications, the two mass model is able to capture many aspects of oscillation features. Small amplitude oscillation appears as a Hopf bifurcation from a steady state as P_s passes a threshold value [9]. More recently, Lucero [12] studied large amplitude oscillations and found coexistence of multiple equilibria and studied their stability and bifurcations.

The two mass model has been widely used as a glottal source for speech synthesis, and many improvements and extensions have been made to incorporate additional physics. For example, see [3] for including effects of flow viscosity and flow separation; see [20], [26], for works on four and multiple mass models and applications. In [2], a continuum elastic system under natural boundary condition is used to find intrinsic vibrational eigenmodes of vocal fold tissues.

Though two mass model is already simple looking, the threshold oscillation condition on minimum lung pressure is still implicit analytically. Titze proposed his celebrated body-cover model in 1988 [22]. The model is based on the hypothesis that during fold oscillation, the fold cover (epithelium and superficial layers of vocal ligament) propagates a surface mucosal wave in the direction of the airflow, and the body (deep layer of the vocal ligament and muscle) is stationary. The fold shape is approximated as a straight line connecting the fold entry of height h_1 and the fold exit of height h_2 . Taylor expanding a mucosal wave with constant velocity c , Titze [22] approximated:

$$h_1 = 2(h_0 + \hat{h} + \tau \hat{h}_t), \quad h_2 = 2(h_0 + \hat{h} - \tau \hat{h}_t), \quad (1.2)$$

where $\tau = 2L/c$ the travel time for the mucosal wave to reach exit from entrance, L half length of glottis along the flow direction, h_0 half glottal width, \hat{h} fold displacement. The fold motion is lumped onto the fold midpoint, and

postulated as:

$$m\hat{h}_{tt} + r\hat{h}_t + k\hat{h} = \frac{1}{2L} \int_{-L}^L P(x) dx \equiv P_g, \quad (1.3)$$

where m , r , and k are the mass, damping, and stiffness of the oscillating portion of the vocal fold about the midpoint, $P(x)$ is the pressure distribution along the flow direction x . Using Bernoulli's law and linear fold shape, [22] showed in particular that for small fold vibration about rectangular fold equilibrium position h_0 :

$$P_g = P_s \left(1 - \frac{h_2}{h_1}\right) = \frac{2\tau P_s \hat{h}_t}{h_0 + \hat{h} + \tau \hat{h}_t}, \quad (1.4)$$

where P_s is the subglottal pressure, and exit pressure is zero. Linearizing (1.4) about h_0 , one has:

$$m\hat{h}_{tt} + (r - 2\tau P_s h_0^{-1})\hat{h}_t + k\hat{h} = 0, \quad (1.5)$$

from which follows the threshold pressure condition setting the effective damping coefficient to zero:

$$P_{s,*} = h_0 r / 2\tau. \quad (1.6)$$

So the oscillation threshold pressure is proportional to damping constant r , and glottal half width h_0 .

Titze model explains in a transparent fashion the formation of oscillation, and provides a very simple onset condition (1.6). It shows how the energy input from the airflow balances the intrinsic fold damping due to the \hat{h}_t dependence of the fold driving force P_g . The agreement of Titze model and the two mass model is discussed in [22], [23]. The onset condition (1.6), especially $P_{s,*}$ being linear in r , is also supported by experiments [24]. For results of Titze model on converging and diverging prephonatory fold shapes, large amplitude oscillations, and hysteresis phenomenon at onset-offset, see [24], [13], [14].

In either the two mass model or Titze model, the vocal fold shape is not described as a continuous curve as we see in physical reality, and that the treatment of air flow is rather crude. It is our goal to seek a more accurate and more first principle based model so that these two aspects are much improved. The fluid characteristics of vocal flow during phonation have been noted in [8], [22], [23], [12], [14], among others, and also recently studied by Pelorson et al [17] using unsteady-flow measurements and visualization techniques. The flow is essentially two dimensional with Mach number on the order of 10^{-1} , and Reynolds number on the order of 10^3 . The flow in the bulk is nearly inviscid except in the viscous boundary layers.

Our starting point is the two dimensional compressible isentropic Navier-Stokes equations for the air flow. Motivated by [8] and [22], we model the vocal fold as a finite mass elastic tube of cross sectional area $A(x, t)$ with elastic attachments (muscles) onto nearby walls (bones). We shall either take rectangular cross section (length $2L$ in x , width $2w$ in z , and variable height $2h$ in y , then $A = 4wh$) or an elliptical shaped cross section (with principal axes of lengths $2h$ in y , and $2w$ in z). The air flows from $x = -L$ to $x = L$, symmetric across $y = 0$, and is independent of z . The purpose of keeping the z dimension is for later reconstruction of a three dimensional picture of fold motion in part II [7]. Assuming that the flow is essentially along x direction, and its variation transverse to the tube is small except in the boundary layers near the folds, we average the flow on each cross section and derive a reduced quasi-one-dimensional system. The motion of fold cover is modeled by a damped driven elastic wave equation, considering the elastic tension on the fold cover and elastic forces in the attached muscles.

The reduced quasi-one-dimensional aerodynamic equations are:

- conservation of mass:

$$(A\rho)_t + (\rho u A)_x = 0, \quad (1.7)$$

ρ air density, u air velocity;

- conservation of momentum:

$$u_t + uu_x = -\frac{1}{\rho}p_x + \frac{A_t u}{A} + \frac{4\mu}{3\rho}A^{-1}(A u_x)_x - \frac{2\mu}{3\rho}A^{-1}A_{tx}, \quad (1.8)$$

p air pressure, μ air viscosity.

The force balance on the fold cover gives:

- the dynamic boundary motion:

$$m(A - A_{eq})_{tt} = \sigma(A - A_{eq})_{xx} - \alpha(A - A_{eq})_t - \beta(A - A_{eq}) + Sp + f_m. \quad (1.9)$$

Here: m is the fold mass density; σ is the longitudinal elastic tension of the fold [6], [18]; α is the muscle damping coefficient; β is an elastic modulus modeling the vibration property of the fold in the transverse; S is a cross section shape factor, $S = 4w$ for rectangular cross sections, and $S = \pi w$ for elliptical cross sections; f_m is a prescribed function to model muscle tone so that a particular fold shape A_{eq} (converging or diverging or flat) serves as an equilibrium state. In general, α , β , σ are functions of x , to model the varying stiffness of fold, since the fold cover is stiffer towards $x = \pm L$.

We shall write (1.9) into:

$$mA_{tt} = \sigma A_{xx} - \alpha A_t - \beta A + Sp + \tilde{f}_m, \quad (1.10)$$

where $\tilde{f}_m = \tilde{f}_m(x)$ is prescribed forcing.

- The equation of state:

$$p = \kappa \rho^\gamma, \quad \gamma > 1, \quad \kappa > 0. \quad (1.11)$$

The system (1.7)-(1.11) is closed, and we solve an initial boundary value problem on $x \in [-L, L]$ with proper in-flow/out-flow boundary conditions and initial fold shape. We shall consider *the inviscid limit of our model* by setting $\mu = 0$ and normalize $m = 1$ while analyzing the oscillation onset in this paper. We shall show via a stability analysis that the linearized system near a flat fold admits (fold cover) waves in A of the form $\sin(kx - \omega t)$, for k

and ω real and nonzero if the glottal airflow has enough energy to overcome fold damping. This is the kind of fold surface mucosal wave hypothesized in Titze model [22]. We also obtain similar onset conditions for small amplitude oscillations due to the occurrence of a pair of imaginary eigenvalues, consistent with early models. Under the quasi-steady approximation as in Titze model, we obtain a single fold equation resembling (1.5), the correction term is nonlocal but does depend on \hat{h}_t , and it plays the role of transferring aerodynamic energy onto the fold.

We remark that the viscous effect is important when the fold is near closure, or when the fold is diverging enough so that flow separation occurs inside the glottis [22], [17], [3]. Our model above has not yet taken this aspect into full consideration. However, since flow separation decreases the pressure (pressure is zero from the point where flow detaches from the fold) towards the exit $x = L$, and the fold is relatively stiff there, we can minimize the effect on fold motion by modeling the increasing stiffness of fold near the exit. Our model captures the compressibility of air flow which is critical during the opening of the folds [16]. The viscous effect during the opening or near closure of vocal fold is delicate and requires further modeling, yet we have not found the lack of it to be a major problem in our simulation of the opening and closing of fold motion in part II [7].

In studying collapsible tubes ([19], [11]) and air flow through duct of spatially varying cross section [25], it is common to use (1.7) with the one dimensional unsteady Euler equation which is (1.8) with the last three terms omitted. The major differences in the two modeling problems are: (1) a vocal fold is fast oscillatory in time (e.g. 100 - 200 Hz), (2) the vocal fold carries mass, and a *dynamic (damped driven wave) equation* is necessary to describe the fold motion; moreover, the vocal fold has mechanical damping. In contrast, collapsible tubes are massless and damping free [19]. The conservation of mass and momentum in the collapsible tube system is recovered when $A_t \ll A$ and $\mu \rightarrow 0$. Though in both collapsible tube and vocal flow

problems, the flow is close to being incompressible, we opt not to make such an approximation as in [19]. This is because an equation of state has to be determined in lieu of the natural equation of state (1.11) when we consider quasi-one-dimensional reduction. For collapsible tubes, see [19] and [11], the tube cross section is related to the pressure p by a tube law: $p = A^{n_1} - A^{n_2}$ with $\rho = \text{constant}$. The tube law needs experimental measurements and depends also on materials involved. Due to lack of our knowledge of whether such a law exists for vocal fold, we choose to use the natural equation of state for air (1.11) and work with compressible flow equations. It is interesting to investigate how good it is to use the incompressible Navier-Stokes equations for the two dimensional flow as a viable alternative. To rephrase our formulation, the ρ and p are related by the equation of state, the gamma gas law (1.11); then the cross section A is related to p *dynamically*.

The rest of the paper is organized as follows. In section 2, we derive the fluid part of the quasi-one-dimensional system (1.7)-(1.11) from the two dimensional isentropic compressible Navier-Stokes equations by averaging solutions over the flow cross section. In section 3, we perform a linear stability analysis near a flat (rectangular) fold and deduce onset condition for the minimum flow energy to ensure the occurrence of a pair of pure imaginary eigenmodes. In section 4, we make the quasi-steady approximation within our model, derive the single equation for the fold cover, obtain an onset condition for small amplitude oscillation, and compare with findings in [22]. Numerical issues, and numerical results will be presented in part II [7].

2 Derivation of Reduced Flow Equations

We derive the fluid part of the model system assuming that the fold varies in space and time as $A = A(x, t)$. Consider a two dimensional slightly viscous subsonic air flow in a channel with spatially temporally varying cross section in two space dimensions, $\Omega_0 = \Omega_0(t) = \{(x, y) : x \in [-L, L], y \in$

$[-A(x, t)/2, A(x, t)/2]\}$, where $A(x, t)$ denotes the channel width with a slight abuse of notation, or cross sectional area since the third dimension is uniform. The two dimensional Navier-Stokes equations in differential form are ([1], page 147):

- conservation of mass:

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0; \quad (2.1)$$

- conservation of momentum:

$$(\rho \vec{u})_t = -\nabla \cdot (\rho (\vec{u} \otimes \vec{u})) + \text{div}(\sigma); \quad (2.2)$$

where σ is the stress tensor, $\sigma = (\sigma_{ij}) = -p\delta_{ij} + d_{ij}$, and:

$$d_{ij} = 2\mu (e_{ij} - \frac{\text{div} \vec{u}}{3} \delta_{ij}), \quad e_{ij} = \frac{1}{2}(u_{i,x_j} + u_{j,x_i}), \quad (x_1, x_2) \equiv (x, y);$$

μ is the fluid viscosity; $\Omega(t)$ is any volume element of the form:

$$\Omega(t) = \{(x, y) : x \in [a, b] \subset [-L, L], y \in [-A(x, t)/2, A(x, t)/2]\}. \quad (2.3)$$

The equation of state is (1.11).

The boundary conditions on (ρ, \vec{u}) are:

- (1) on the upper and lower boundaries $y = \pm A(x, t)/2$, $\rho_y = 0$, and $\vec{u} = (0, \pm A_t/2)$, the velocity no slip boundary condition;
- (2) at the inlet, $\vec{u}(-L, y, t) = \vec{u}_l$, a prescribed inlet velocity, $\rho(-L, y, t) = \rho_l$, a prescribed inlet density (deduced from input pressure);
- (3) at the outlet, $\vec{u}_x(L, y, t) = 0$.

We are only concerned with flows that are symmetric in the vertical. For positive but small viscosity, the flows are laminar in the interior of Ω_0 and form viscous boundary layers near the upper and lower edges. The vertically averaged flow quantities are expected to be much less influenced by the boundary layer behavior as long as $A(x, t)$ is much larger than $O(\mu^{1/2})$. We also ignore effects of possible flow separation inside Ω_0 when it becomes divergent with large enough opening.

Let us assume that the flow variables obey:

$$\begin{aligned} |u_{1,y}| &\ll |u_{1,x}|, \quad |u_{2,y}| \ll |u_{1,x}|, \quad \text{away from boundaries of } \Omega_0, \\ |\vec{u}_y| &\gg |\vec{u}_x|, \quad \text{near the boundaries of } \Omega_0, \\ |\rho_y| &\ll |\rho_x|, \quad \text{throughout } \Omega_0. \end{aligned} \quad (2.4)$$

These are consistent with physical observations in the viscous boundary layers ([1], page 302), namely, there are large vertical velocity gradients, yet small pressure or density gradients in the boundary layers. The boundary layers are of width $O(\mu^{1/2})$. Denote by $\bar{\rho}$, \bar{u}_1 , the vertical averages of ρ and u_1 . Note that the exterior normal $\vec{n} = (-A_x/2, 1)/(1 + A_x^2/4)^{1/2}$ if $y = A/2$, $\vec{n} = (-A_x/2, -1)/(1 + A_x^2/4)^{1/2}$ if $y = -A/2$.

Let $a = x$, $b = x + \delta x$, $\delta x \ll 1$, t slightly larger than t_0 . We have:

$$\frac{d}{dt} \int_{\Omega(t)} \rho dV = \frac{d}{dt} \int_{\Omega(t_0)} \rho J(t) dV = \int_{\Omega(t_0)} \rho_t J(t) dV + \int_{\Omega(t_0)} \rho J_t dV, \quad (2.5)$$

where $J(t)$ is the Jacobian of volume change from a reference time t_0 to t . Since $\Omega(t)$ is now a thin slice, $J(t) = \frac{A(t)}{A(t_0)}$ for small δx , and $J_t = A_t(t)/A(t_0)$. The second integral in (2.5) is:

$$\int_{\Omega(t_0)} \rho J_t dV = \bar{\rho} \frac{A_t(t)}{A(t_0)} A(t_0) \delta x = \bar{\rho} A_t(t) \delta x. \quad (2.6)$$

The first integral is simplified using (2.1) as:

$$\int_{\Omega(t_0)} \rho_t J(t) dV = \int_{\Omega(t)} \rho_t dV = - \int_{\partial\Omega(t)} \rho \vec{u} \cdot \vec{n} dS. \quad (2.7)$$

We calculate the last integral of (2.7) further as follows:

$$\begin{aligned} \int_{\partial\Omega} \rho \vec{u} \cdot \vec{n} ds &= \int_{-A/2}^{A/2} (-\rho u_1)(x, y, t) dy + \int_{-A/2}^{A/2} (\rho u_1)(x + \delta x, y, t) dy \\ &+ \int_x^{x+\delta x} \rho \cdot (0, A_t/2) \cdot (-A_x/2, 1) dx \\ &+ \int_x^{x+\delta x} \rho \cdot (0, -A_t/2) \cdot (-A_x/2, -1) dx \end{aligned}$$

$$\begin{aligned}
&= \overline{\rho u_1} A|_x^{x+\delta x} + \frac{\delta x}{2}(\rho A_t)|_{y=A/2} + \frac{\delta x}{2}(\rho A_t)|_{y=-A/2} + O((\delta x)^2) \\
&\approx (\bar{\rho} \cdot \bar{u}_1 A)|_x^{x+\delta x} + \bar{\rho} A_t \delta x + O((\delta x)^2),
\end{aligned} \tag{2.8}$$

where we have used the smallness of ρ_y to approximate $\rho|_{y=\pm A/2}$ by $\bar{\rho}$ and $\overline{\rho u_1}$ by $\bar{\rho} \cdot \bar{u}_1$.

Combining (2.5)-(2.7), (2.8) with:

$$\frac{d}{dt} \int_{\Omega} \rho dV = -(\bar{\rho} \cdot \bar{u}_1 A)|_x^{x+\delta x} + O((\delta x)^2), \tag{2.9}$$

dividing by δx and sending it to zero, we have:

$$(\bar{\rho} A)_t + (\bar{\rho} \cdot \bar{u}_1 A)_x = 0,$$

which is (1.7).

Next consider $i = 1$ in the momentum equation, $a = x$, $b = x + \delta x$. We have similarly with (2.2):

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega(t)} \rho u_1 dV = \int_{\Omega(t)} (\rho u_1)_t dV + \int_{\Omega(t_0)} \rho u_1 J_t dV \\
&= - \int_{\partial\Omega(t)} \rho u_1 \vec{u} \cdot \vec{n} dS + \int_{\partial\Omega(t)} \sigma_{1,j} \cdot \vec{n}_j dS + \bar{\rho} \bar{u}_1 A_t \delta x + O(\delta x^2).
\end{aligned} \tag{2.10}$$

We calculate the integrals of (2.10) below.

$$\frac{d}{dt} \int_{\Omega} \rho u_1 dV = (\overline{\rho u_1} A)_t \delta x + O((\delta x)^2) \approx (\bar{\rho} \cdot \bar{u}_1 A)_t \cdot \delta x + O((\delta x)^2), \tag{2.11}$$

where $u_1 = 0$ on the upper and lower boundaries is used.

$$\int_{\partial\Omega} \rho u_1 \vec{u} \cdot \vec{n} dS = (\bar{\rho} \cdot \bar{u}_1^2 A)|_x^{x+\delta x} + O(\delta x \mu^{1/2}), \tag{2.12}$$

where the smallness of $u_{1,y}$ in the interior and small width of boundary layer $O(\mu^{1/2})$ gives the $O(\mu^{1/2})$ for approximating $\overline{u_1^2}$ by $\bar{u}_1 \cdot \bar{u}_1$.

$$\begin{aligned}
\int_{\partial\Omega} -p \delta_{1,j} n_j dS &\approx -\bar{p} A|_x^{x+\delta x} + \int_x^{x+\delta x} p A_x dx \\
&= -\bar{p} A|_x^{x+\delta x} + \bar{p} A_x \delta x + O((\delta x)^2).
\end{aligned}$$

Noticing that:

$$d_{11} = 2\mu(u_{1,x} - (u_{1,x} + u_{2,y})/3), \quad d_{12} = 2\mu(u_{1,y} + u_{2,x}).$$

It follows that

$$\overline{d_{11}} = \frac{4}{3}\mu\overline{u_{1,x}} - \frac{2\mu A_t}{3A}.$$

Thus the contribution from the left and right boundaries located at x and $x + \delta x$ is:

$$\sum_{l,r} \int_{l,r} d_{11} n_1 = A \overline{d_{11}}|_x^{x+\delta x} = \frac{4}{3}A\mu\overline{u_{1,x}}|_x^{x+\delta x} - \frac{2\mu A_t}{3}|_x^{x+\delta x}. \quad (2.13)$$

The contribution from the upper and lower boundaries is:

$$\begin{aligned} \sum_{\pm} \int_{y=\pm A/2} d_{11} n_1 dS &= -d_{11} A_x \delta x / 2|_{y=A/2} - d_{11} A_x \delta x / 2|_{y=-A/2} \\ &= \mu \delta x \sum_{\pm} O(\partial_y \vec{u})|_{y=\pm A/2}. \end{aligned} \quad (2.14)$$

Similarly,

$$\sum_{\pm} \int_{y=\pm A/2} d_{12} n_2 dS = \mu \delta x \sum_{\pm} O(\partial_y \vec{u})|_{y=\pm A/2}. \quad (2.15)$$

Since $\partial_y \vec{u}|_{y=\pm A/2} = O(\mu^{-1/2})$, the viscous flux from the boundary layers are $O(\delta x \mu^{1/2})$, much larger than the averaged viscous term $\delta x \frac{4\mu}{3}(A\overline{u_{1,x}})_x = O(\delta x \mu)$. We notice that the vertically averaged quantities have little dependence on the viscous boundary layers unless A is on the order $O(\mu^{1/2})$. Hence the quantities from upper and lower edges in (2.14) and (2.15), and that in (2.12), should balance themselves. Omitting them altogether, and combining remaining terms that involve only $\overline{u_1}$, $\overline{\rho}$ in the bulk, we end up with (after dividing by δx and sending it to zero):

$$(\overline{\rho} \cdot \overline{u_1} A)_t + (\overline{\rho} \cdot \overline{u_1}^2 A)_x = -(\overline{\rho} A)_x + A_x \overline{\rho} + \overline{\rho} \overline{u_1} A_t + \frac{4\mu}{3}(A\overline{u_{1,x}})_x - 2\mu A_{tx} / 3. \quad (2.16)$$

Simplifying (2.16) with the continuity equation (1.7), we find that:

$$\overline{u_{1t}} + \overline{u_1} \overline{u_{1x}} = -\overline{\rho}_x / \overline{\rho} + \frac{A_t \overline{u_1}}{A} + \frac{4\mu}{3\overline{\rho}} A^{-1} (A\overline{u_{1,x}})_x - \frac{2\mu A^{-1} A_{tx}}{3\overline{\rho}}, \quad (2.17)$$

which is (1.8).

3 Linear Stability Analysis near a Flat Fold

In this section, we discuss the existence of a neutral oscillation mode of the linearized system around a flat fold in the limit $\mu \rightarrow 0$. Such a mode provides an onset condition of oscillation, see similar analysis for the Titze model in [22] and [14]. It turns out that such a mode exists under a threshold condition for system (1.7)-(1.11) because of the $A_t u/A$ term. In our case, it is essential that the derivation originated with the no slip boundary condition on the fold, which allows enough energy of the background flow to transfer to the fold to offset the loss there. This energy transfer mechanism is expounded in Titze [22] in his body-cover model. We show with a stability analysis that our model system supports such a physical picture of flow induced oscillation.

We assume that the cross section is rectangular. The system (1.7)-(1.11) admits constant steady states: (u_0, p_0, A_0) , $p_0 = \kappa \rho_0^\gamma$, satisfying:

$$-\beta A_0 + 4wp_0 = f_m, \quad (3.1)$$

where f_m is a constant so that A_0 matches the height of the connecting vocal tract. We are interested in conditions leading to the small amplitude oscillations near the constant steady states. This is similar to Titze [22], where a lumped ODE is proposed and analyzed for the fold center using mucosal wave approximation. However here, we calculate directly from (1.7)-(1.11), and do not make any further modeling assumptions. Since we have zero background pressure gradient, our results do not compare directly with [22], though qualitative features remain. Another comparison with [22] is performed in the next section under quasi-steady approximation where small pressure gradient is present.

Letting $u = u_0 + \hat{u}$, $p = p_0 + \hat{p}$, $\rho = \rho_0 + \hat{\rho}$, $A = A_0 + \hat{A}$, and linearizing (1.7)-(1.11) with $\mu = 0$, we get:

$$(A_0 \hat{\rho} + \hat{A} \rho_0)_t + (\rho_0 A_0 \hat{u} + \rho_0 u_0 \hat{A} + u_0 A_0 \hat{\rho})_x = 0, \quad (3.2)$$

$$\hat{u}_t + u_0 \hat{u}_x + \frac{1}{\rho_0} \hat{p}_x = \frac{u_0}{A_0} \hat{A}_t, \quad (3.3)$$

$$\hat{A}_{tt} = \sigma \hat{A}_{xx} - \alpha \hat{A}_t - \beta \hat{A} + 4w\hat{p}. \quad (3.4)$$

Equations (3.2)-(3.3) are written as:

$$A_0 \hat{\rho}_t + \rho_0 \hat{A}_t + \rho_0 A_0 \hat{u}_x + \rho_0 u_0 \hat{A}_x + u_0 A_0 \hat{\rho}_x = 0, \quad (3.5)$$

$$\hat{u}_t + u_0 \hat{u}_x + \frac{\hat{p}_x}{\rho_0} = \frac{u_0}{A_0} \hat{A}_t. \quad (3.6)$$

Applying the operator $\partial_t + u_0 \partial_x$ on (3.5) and using (3.6), we find:

$$(\partial_t + u_0 \partial_x)(A_0 \hat{\rho}_t + \rho_0 \hat{A}_t + \rho_0 u_0 \hat{A}_x + u_0 A_0 \hat{\rho}_x) - A_0 \hat{p}_{xx} + u_0 \rho_0 \hat{A}_t = 0. \quad (3.7)$$

Differentiating (1.11) gives: $p_t = \kappa \gamma \rho^{\gamma-1} \rho_t$, or $\hat{p}_t = \kappa \gamma \rho_0^{\gamma-1} \hat{\rho}_t$ upon linearizing at $\rho = \rho_0$. Similarly, $\hat{p}_x = \kappa \gamma \rho_0^{\gamma-1} \hat{\rho}_x$. With these relations, equation (3.7) becomes:

$$\begin{aligned} & \Gamma(\partial_t + u_0 \partial_x)^2 \hat{p} - \hat{p}_{xx} + \frac{u_0 \rho_0}{A_0} \hat{A}_t \\ & + A_0^{-1} \rho_0 (\partial_t + u_0 \partial_x)^2 \hat{A} = 0, \end{aligned} \quad (3.8)$$

where:

$$\Gamma = \frac{1}{\kappa \gamma \rho_0^{\gamma-1}} = \frac{1}{c^2},$$

with c being the speed of sound at air density ρ_0 .

Applying the operator $A_0 \Gamma (\partial_t + u_0 \partial_x)^2 - A_0 \partial_{xx}$ to both sides of (3.4), we get:

$$\begin{aligned} & A_0 [\Gamma \partial_{tt} + 2\Gamma u_0 \partial_{xt} + (\Gamma u_0^2 - 1) \partial_{xx}] \cdot (\hat{A}_{tt} - \sigma \hat{A}_{xx} + \alpha \hat{A}_t + \beta \hat{A}) \\ & = -4w\rho_0 [(\partial_t + u_0 \partial_x)^2 \hat{A} + u_0 \hat{A}_t]. \end{aligned} \quad (3.9)$$

Substituting the mode $\hat{A} = A_m e^{im'x + \lambda t}$, $m' = \frac{m\pi}{L}$, we end up with the following algebraic equation of degree four for λ :

$$\begin{aligned} & [\Gamma \lambda^2 + 2\Gamma u_0 m' \lambda i + (1 - \Gamma u_0^2) m'^2] \cdot [\lambda^2 + \sigma m'^2 + \alpha \lambda + \beta] \\ & = -4w\rho_0 A_0^{-1} [\lambda^2 + 2u_0 i m' \lambda - u_0^2 m'^2] - \frac{4w u_0 \rho_0 \lambda}{A_0}, \end{aligned} \quad (3.10)$$

or:

$$\begin{aligned}
& \Gamma\lambda^4 + (\alpha\Gamma + 2\Gamma u_0 m' i)\lambda^3 + \\
& (\Gamma(\beta + \sigma m'^2) + 2\alpha\Gamma u_0 m' i + 4w\rho_0 A_0^{-1} + (1 - \Gamma u_0^2)m'^2)\lambda^2 \\
& + (2\Gamma u_0 m' i(\beta + \sigma m'^2) - \alpha m'^2(\Gamma u_0^2 - 1) \\
& + 4w\rho_0 A_0^{-1}(2u_0 i m') + \frac{4wu_0\rho_0}{A_0})\lambda \\
& + [(\beta + \sigma m'^2)(1 - \Gamma u_0^2)m'^2 + 4w\rho_0 A_0^{-1}(-u_0^2 m'^2)] = 0. \tag{3.11}
\end{aligned}$$

Proposition 3.1 *If*

$$\rho_0 u_0^2 > \alpha(\rho_0 u_0 + \Gamma\beta\frac{u_0 A_0}{4w}), \tag{3.12}$$

(3.11) has a pair of pure imaginary solution $\lambda = \pm i\eta$, $\eta \neq 0$ being real, which implies the existence of a pair of oscillatory modes to the linearized system (3.2)-(3.4) of the form $e^{\pm(im\pi x/L + i\eta t)}$, for real and nonzero numbers m and η . The transition into oscillation is a Hopf bifurcation.

Proof: Let $\lambda = i\eta$ in (3.11), where η is real. The real and imaginary parts give respectively:

$$\begin{aligned}
& \Gamma\eta^4 + 2\Gamma u_0 m' \eta^3 - [\Gamma(\sigma m'^2 + \beta) + m'^2(1 - \Gamma u_0^2) + \frac{4w\rho_0}{A_0}]\eta^2 \\
& + [-2\Gamma u_0 m'(\beta + \sigma m'^2) - \frac{8w\rho_0 u_0 m'}{A_0}]\eta \\
& + [m'^2(1 - \Gamma u_0^2)(\sigma m'^2 + \beta) - \frac{4u_0^2 m'^2 w\rho_0}{A_0}] = 0, \tag{3.13}
\end{aligned}$$

and:

$$-\alpha\Gamma\eta^3 - 2\alpha\Gamma u_0 m' \eta^2 + m'^2\alpha(1 - \Gamma u_0^2)\eta + \frac{4wu_0\rho_0\eta}{A_0} = 0. \tag{3.14}$$

For $\eta \neq 0$, $\alpha \neq 0$, we have from (3.14):

$$\Gamma\eta^2 + 2\Gamma u_0 m' \eta - m'^2(1 - \Gamma u_0^2) - \frac{4wu_0\rho_0}{A_0\alpha} = 0,$$

so:

$$\eta = -u_0 m' \pm c \sqrt{m'^2 + 4wu_0\rho_0/(A_0\alpha)}. \quad (3.15)$$

Now we regard the left hand side of (3.13) as a continuous function of m' , call it $F(m')$. For $|m'| \gg 1$, $\eta \sim (-u_0 \pm c)m'$, direct calculation shows:

$$F(m') \sim -\frac{4w\rho_0}{\Gamma A_0} m'^2 < 0.$$

While for $|m'| \ll 1$,

$$\eta \sim \pm 2w \sqrt{\frac{u_0\rho_0}{\Gamma A_0\alpha}} + O(m'),$$

and:

$$F(m') \sim \frac{4wu_0\rho_0}{\Gamma A_0\alpha} \left(\frac{4wu_0\rho_0}{A_0\alpha} - \Gamma\beta - \frac{4w\rho_0}{A_0} \right) > 0,$$

provided:

$$\frac{4wu_0\rho_0}{A_0} > \alpha \left(\frac{4w\rho_0}{A_0} + \Gamma\beta \right), \quad (3.16)$$

holds, which is just (3.12). Under (3.12), $F(m) = 0$ has a nonzero real solution, hence an oscillatory mode solution exists to (3.2)-(3.4). Finally, noticing that equations (3.13)-(3.14) are invariant under the symmetry transform $(\eta, m') \rightarrow (-\eta, -m')$, we conclude that the oscillatory modes exist as a pair, and oscillation appears as a Hopf bifurcation.

Remark 3.1 *Condition (3.12) says that the fluid energy must be large enough to overcome the fold damping due to α . Without the lower order term $A_t u/A$, the same calculation would show that $F(m') = -4w\rho_0 m'^2/(\Gamma A_0) < 0$ for all m' , implying non-existence of oscillatory mode. Condition (3.16) is similar to the threshold pressure in Titze's model [22] in the sense that minimum energy (analogous to minimum lung pressure) is proportional to the fold damping coefficient and the prephonatory half width A_0 .*

Remark 3.2 *There is another neutral yet nonoscillatory mode from (3.13)-(3.14), namely, $\eta = 0$, and:*

$$\sigma|m'_0|^2 = \frac{4u_0^2 w \rho_0}{(1 - M^2)A_0} - \beta, \quad (3.17)$$

provided the right hand side expression is positive. As we turn on $\mu > 0$ but small, the oscillation mode in Proposition 2.1 will generically be slightly perturbed and yet preserves its oscillatory nature. The calculations are tedious and not shown here.

4 Quasi-steady Approximation and Relations with the Titze Model

The glottal flow is often regarded as nearly quasisteady and inviscid in the bulk [17], and the Bernoulli's law is adopted as an approximation, [8], [22] among others. In this approximation, the temporal variation of flow variables is considered much slower than that of the fold motion. Now let us drop the time derivatives, and viscous term in the flow equations to get:

$$(\rho u A)_x = 0, \quad (4.1)$$

$$u u_x = -\frac{p_x}{\rho} + \frac{A_t u}{A}, \quad (4.2)$$

while keeping the fold dynamic equation and the equation of state the same.

Our goal is to derive a closed equation for the cross section area A . Integrating (4.1) and (4.2) in x using the equation of state shows:

$$\rho u A = Q_0, \quad (4.3)$$

$$u^2/2 + \frac{\gamma \rho^{\gamma-1}}{\gamma-1} = \int_{-L}^x \frac{A_t u}{A} dx + P_0, \quad (4.4)$$

where P_0 and Q_0 are constants determined by the flow conditions at the inlet $x = -L$. Note that without the integral term with $\frac{A_t u}{A}$, (4.4) becomes the standard Bernoulli's law.

Substituting (4.3) into (4.4) gives:

$$\frac{Q_0^2}{2\rho^2 A^2} + \frac{\gamma\rho^{\gamma-1}}{\gamma-1} = Q_0 \int_{-L}^x \frac{A_t}{\rho A^2} dx + P_0. \quad (4.5)$$

We would obtain a closed equation on A if (4.5) could be solved for ρ in terms of A .

Suppose that A undergoes small vibration about a constant state A_0 which satisfies:

$$\frac{Q_0^2}{2\rho_0^2 A_0^2} + \frac{\gamma\rho_0^{\gamma-1}}{\gamma-1} = P_0,$$

then (4.5) can be solved for ρ by perturbation. Letting $A = A_0 + \hat{A}$, $\rho = \rho_0 + \hat{\rho}$, we find:

$$-\frac{Q_0^2}{\rho_0^3 A_0^2} \hat{\rho} - \frac{Q_0^2}{\rho_0^2 A_0^2} \hat{A} + \gamma\rho_0^{\gamma-2} \hat{\rho} - Q_0 \int_{-L}^x \frac{\hat{A}_t}{A_0^2 \rho_0} = 0,$$

or:

$$\hat{\rho} \cdot \left(\gamma\rho_0^{\gamma-2} - \frac{Q_0^2}{\rho_0^3 A_0^2} \right) = \frac{Q_0^3}{A_0^2 \rho_0} \hat{A} + \frac{Q_0}{A_0^2 \rho_0} \int_{-L}^x \hat{A}_t. \quad (4.6)$$

If $\gamma\rho_0^{\gamma-2} > \frac{Q_0^2}{\rho_0^3 A_0^2}$, or:

$$\rho_0^{\gamma+1} > \frac{Q_0^2}{\gamma A_0^2}, \quad (4.7)$$

which means large pressure for given Q_0 and A_0 , then:

$$\hat{\rho} = \frac{Q_0^2}{\rho_0 A_0^2} \cdot \frac{1}{\gamma\rho_0^{\gamma-2} - \frac{Q_0^2}{\rho_0^3 A_0^2}} \left[\int_{-L}^x \hat{A}_t + \frac{Q_0}{\rho_0 A_0} \hat{A} \right]. \quad (4.8)$$

Now substituting (4.8) into the fold dynamic A equation, we obtain:

$$\hat{A}_{tt} - \sigma \hat{A}_{xx} + \alpha \hat{A}_t + \beta \hat{A} = c_{aero} \left[\int_{-L}^x \hat{A}_t + \frac{Q_0}{\rho_0 A_0} \hat{A} \right], \quad (4.9)$$

where:

$$c_{aero} = \frac{4w\kappa\gamma\rho_0^{\gamma-1}}{\gamma\rho_0^{\gamma-2} - \frac{Q_0^2}{\rho_0^3 A_0^2}} \frac{Q_0^2}{\rho_0 A_0^2} > 0. \quad (4.10)$$

Equation (4.9) can be written as:

$$\hat{A}_{tt} - \sigma \hat{A}_{xx} + (\alpha \hat{A} - c_{aero} \int_{-L}^x \hat{A})_t + (\beta - c_{aero} Q_0 \rho_0^{-1} A_0^{-1}) \hat{A} = 0. \quad (4.11)$$

Equation (4.11) is a linear wave equation with damping and pumping. This is easy to see from the energy identity. Assuming that $\hat{A}_x(\pm L, t) = 0$, we have:

$$\begin{aligned} \frac{d}{dt} \int_{-L}^L dx \left(\frac{\sigma}{2} \hat{A}_x^2 + \frac{1}{2} \hat{A}_t^2 + (\beta - c_{aero} Q_0 \rho_0^{-1} A_0^{-1}) \hat{A}^2 / 2 \right) = \\ -\alpha \int_{-L}^L dx \hat{A}_t^2 + c_{aero} \left(\int_{-L}^L dx \hat{A}_t \right)^2 / 2. \end{aligned} \quad (4.12)$$

We shall require that:

$$\beta - c_{aero} Q_0 \rho_0^{-1} A_0^{-1} > 0, \quad (4.13)$$

which is true if ρ_0 is sufficiently large for fixed β , A_0 , and Q_0 . Then (4.12) says that the rate of change of the total energy in time depends on the balance of the natural fold damping (the negative term with prefactor α) and the energy input from the aerodynamic flow (the positive term with prefactor c_{aero}).

We see from (4.12) that spatially sinusoidal modes like $e^{im'x}$ will render $\int_{-L}^L \hat{A}_t dx = 0$, hence they do not sustain lossless temporal oscillations. However, there are lossless oscillatory solutions with monotone spatial profiles. To show a simple solution of this kind, extend the incoming flow uniformly to $-\infty$ and modify the energy transfer term to $\int_{-\infty}^x \frac{\hat{A}_t u}{A}$ in (4.11). Let us keep the exit location still at $x = L$. Then seek $\varphi = e^{\nu x} \psi(t)$, $\nu > 0$, $\varphi_x = \hat{A}$, $\psi(t)$ satisfies:

$$\nu \psi_{tt} + (\alpha \nu - c_{aero}) \psi_t + \nu (\beta - c_{aero} Q_0 \rho_0^{-1} A_0^{-1} - \sigma \nu^2) \psi = 0.$$

Since ν^{-1} is the length scale of spatial decay, ν^{-1} is of order $O(L)$. The critical condition for temporal oscillation is $\alpha \nu = c_{aero}$, to remove the damping, or

$$c_{aero} \geq \frac{\alpha}{L},$$

or:

$$4w\kappa Q_0^2 \geq \frac{\alpha A_0^2}{L}.$$

The temporal oscillation frequency is $\sqrt{\beta - c_{aero} Q_0 \rho_0^{-1} A_0^{-1} - \sigma \nu^2}$, which is positive for large ρ_0 and small σ . Notice that the fold positions at $\pm L$: $A_0 + \nu e^{-\nu L} \psi(t)$, $A_0 + \nu e^{\nu L} \psi(t)$, are not equal, they are above and below A_0 simultaneously. When they are above A_0 , the fold is divergent (positive slope along the flow direction); and when they are below A_0 , the fold is convergent (negative slope along the flow direction). However, the two end points are never completely out of phase, i.e, when one is above and the other is below A_0 .

The oscillatory solutions are of very different forms with or without making the quasi-steady approximation, which may underestimate the amount of the energy transfer from air flow onto the vocal fold. Nevertheless, the above spatially monotone solutions under the quasi-steady approximation agree qualitatively with those of the Titze model [22] and shows explicitly the effect of the energy transfer term $(\int_{-L}^x \frac{A_t u}{A})$. This term is nonlocal and different from the local term in Titze model [22], yet is of the same origin and reflects the changing directions of the effective air driving force on open and closing cycles, a key observation of the Titze model.

Both our model and the Titze model capture the positive energy feedback from air flow into the fold motion, though they are not of the same form. Our model has the advantage of a systematic treatment and of characterizing the entire fold shape in the continuum.

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